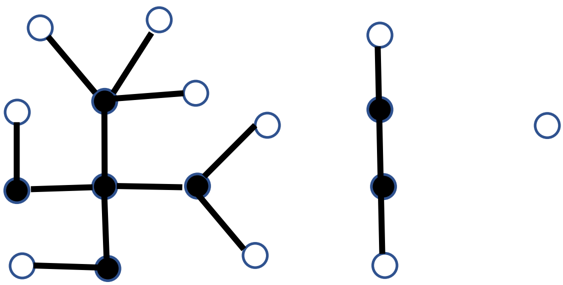
3. TREES

§3.1. Forests

A **forest** is a graph with no cycles. A tree is a connected forest, that is, a forest with just one tree. A **leaf** in a tree is a vertex of degree 0 or 1. Graph Theory, like a lot of branches of mathematics, tends to use homely, familiar words.

Example 1: The following is a forest with two components, each of which is a tree. The leaves are marked as



Theorem 1: The following are equivalent for a graph G with V vertices, E edges and C components.

(1) G is a forest.

(2) Every two vertices in G are joined by at most one unique path.

(3) V = E + C.

Proof:

(1) → (2):

Suppose G is a forest.

Suppose u is joined to v by two different paths:

P_1: u = a_1 a_2 … a_r = v and

P_2: u = b_1 b_2 … b_s = v.

Let a_h be the first vertex such that $a_h \neq b_h$ (there is one such vertex because $P_1 \neq P_2$.

Let a_k be the first vertex with k > h with a_k in P_2 (it exists because $a_r = b_s$).

Suppose $a_k = b_t$.

Then a_h a_{h+1} … a_k b_{t-1} b_{t-2} … b_h b_{h-1} a_h is a cycle, a contradiction.

(2) → (3):

Suppose that every two vertices in G is joined by at most one path.

We prove that V = E + C by induction on E.

If E = 0 then clearly V = C and so the result holds.

Suppose E ≥ 1.

Remove any edge u – v from G, creating the graph H.

Since there is at most one path from u to v in G, u and v are in different components in H.

The component in G that contained u and v now splits into two components, those vertices joined to u and those joined to v.

Hence H has $E - 1$ edges and $C + 1$ components, and so by induction. $V = (E - 1) + (C + 1)$ and hence $V = E + C$.

(3) → (1):

Suppose that $V = E + C$.
If G has a vertices of degree 0 then remove those vertices. The resulting graph will satisfy the equation $V = E + C$. So we may assume that G has no vertex of degree 0.

If G has no vertex of degree 1 then every vertex has degree at least 2.
Then $2E = \Sigma\text{degrees} \geq 2V = 2E + C$, a contradiction.
Hence G has a vertex of degree 1.
We prove that G is a forest by induction on E.
If $E = 0$ then clearly G is a forest.
Suppose $E \geq 1$.
Remove a vertex of degree 1, and the corresponding edge, producing the graph H.
Now H has $V - 1$ vertices and $E - 1$ edges but it still has C components.
Since $V - 1 = E - 1 + C$, H is a forest and so has no cycles.
The edge that was removed cannot be part of a cycle in G and so G is a forest. 🖐☺

§3.2. Centres of Trees

Theorem 2: If G is a tree then $|Z(G)| \leq 2$.

Proof: Let H be the tree obtained by removing all the leaves of G, and their corresponding edges.

Then the eccentricity of a vertex in H is one less than its eccentricity in G.

Hence Z(H) = Z(G).

Continuing in this way we end up with a single vertex, or two adjacent vertices, that is, K_1 or K_2. ✋☺

Corollary: If G is a tree and Z(G) consists of two vertices, they are adjacent.

Example 2: If G is the tree in example then H will be the following tree. Clearly, repeating the process of removing leaves one more time, we will obtain the white vertex as the centre.